Dynamic Beam Model with Internal Damping Rotatory Inertia and Shear Deformation

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A simple and straightforward technique that makes use of the state space method of modern control theory, to include the effects due to internal damping, rotatory inertia, and shear deformation in the dynamic analysis of beams is proposed. It is shown how the resulting beam model which is valid even for high frequencies of vibration, can be used in design situations such as vibration control using linear dampers and dynamic absorbers. A method to include hysteritic internal damping as well as viscoelastic damping is indicated. A numerical example is given to illustrate the use of the proposed beam model in an elementary problem.

Nomenclature

A	= beam cross-sectional area
c	= stress rate coefficient in SLS model
C_k	= damping constant of kth damper or absorber
$\delta(x)$	= Dirac delta function
E	= Young's modulus of elasticity
E^*	= strain rate coefficient in SLS model
ϵ	= normal strain
$f(x,t)$	= distributed load on beam per unit length
ϕ	= angle of rotation of beam cross section
g	= acceleration due to gravity
G	= shear modulus
I	= second moment of area of beam cross section about its neutral axis
k	= Timoshenko shear coefficient
K_k	= spring constant of kth dynamic absorber
ℓ	= beam length
m_k	= kth absorber mass
M	= bending moment at beam cross section
$P_i(t)$	= generalized force coordinate of $Y_i(x)$
Q	= transverse shear force on beam cross section
ρ	= absolute density of beam material
s_k	= displacement of m_k
t	= time
$T_i(t)$	= generalized displacement coordinate of $Y_i(x)$
θ	= shear strain
v	= distance from neutral axis of elemental area dA of beam cross section
x	= position along the beam from one end
$y(x,t)$	= beam deflection
$Y_i(x)$	= ith eigenfunction of beam vibration
σ	= normal stress

Superscripts

$(\dot{\ })$	= derivative with respect to t
$(\ )'$	= derivative with respect to x
$[\ ]^T$	= matrix or vector transposition

Introduction

THE Bernoulli-Euler theory, which is extensively used in the analysis of dynamic systems that can be approximated by beams, neglect the important effects of internal damping of the material, deformation due to shear, and rotatory inertia. Inclusion of these effects increases the order with respect to time of the partial differential equation of beam dynamics from two to five, when Standard Linear Solid (SLS) model[1] is used to represent internal damping. Great effort is needed to solve this equation even for the most simple special case when standard techniques that are analogous to those used for the Bernoulli-Euler equation solution are employed.

Baker, Woolam, and Young[2] have employed energy techniques for preliminary analysis of the Bernouli-Euler (BE) beam with internal damping. De Silva[3] has suggested modern control techniques to analyze the simply supported undamped Timoshenko beam. In the present work a general model with internal damping, shear deformation, and rotatory inertia effects, is derived for the transverse dynamics of beams with general end conditions. The usefulness of the model in treating the beam with vibration controls, such as linear dampers and dynamic absorbers, is shown. A numerical example is given to illustrate the manipulation of the model using available computer programs.

Development of the Beam Model

Timoshenko[4] introduced shear deformation and rotatory inertia effects to the BE equation. For forced vibration the appropriate relations are
shear force-strain relation

$$Q = -kGA\theta \tag{1}$$

equation of transverse motion

$$f(x,t) - \frac{\partial Q}{\partial X} = \rho A \frac{\partial^2 y}{\partial t^2} \tag{2}$$

equation of rotatory motion

$$\frac{\partial M}{\partial x} - Q = \rho I \frac{\partial^2 \phi}{\partial t^2} \tag{3}$$

normal strain-slope relation

$$\epsilon = v \frac{\partial c}{\partial x} \tag{4}$$

and the following self-explanatory relations

$$M = \int_A \sigma v \mathrm{d}A \tag{5}$$

$$I = \int_A v^2 \mathrm{d}A \tag{6}$$

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$$\frac{\partial y}{\partial x} = \phi + \theta \tag{7}$$

The Kelvin-Voigt (KV) model[1] for internal damping does not increase the order with respect to time of the beam equation. More accurate SLS model increases the order by one in both BE and Timoshenko equations. It will be clear that this increases the order of the beam model derived in the present work, by one per mode of vibration. Even though it has been shown[5] that KV and SLS models are almost equivalent for low frequencies of vibration, SLS model is used in the proposed beam model because the Timoshenko effects (shear deformation and rotatory inertia) are significant only at relatively high frequencies. This selection makes the beam model valid even for high frequencies of vibration.

The normal stress-strain relation corresponding to SLS model is

$$\sigma + c\frac{d\sigma}{dt} = E\epsilon + E^*\frac{d\epsilon}{dt} \tag{8}$$

By straightforward manipulation of relations (1)-(8) the following beam equation may be obtained.

$$\begin{aligned} &EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} + c\rho A\frac{\partial^3 y}{\partial t^3} + \frac{\rho^2 I}{kG}\frac{\partial^4 y}{\partial t^4} \\ &+ \frac{c\rho^2 I}{kG}\frac{\partial^5 y}{\partial t^5} + E^* I\frac{\partial^5 y}{\partial x^4 \partial t} - \left(\rho I + \frac{\rho I E}{kG}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} \\ &- \left(c\rho I + \frac{\rho I E^*}{kG}\right)\frac{\partial^5 y}{\partial x^2 \partial t^3} = f + c\frac{\partial f}{\partial t} \\ &+ \frac{\rho I}{kGA}\frac{\partial^2 f}{\partial t^2} + \frac{c\rho I}{kGA}\frac{\partial^3 f}{\partial t^3} - \frac{EI}{kGA}\frac{\partial^2 f}{\partial x^2} - \frac{E^* I}{kGA}\frac{\partial^3 f}{\partial x^2 \partial t} \end{aligned} \tag{9}$$

We assume a normal mode solution to Eq. (9) of the form

$$y(x,t) = \sum_i Y_i(x) \cdot T_i(t) \tag{10}$$

Also the forcing function may be expanded as

$$f(x,t) = \sum_i Y_i(x) \cdot \rho_i(t) \tag{11}$$

Here $T_i(t)$ and $P_i(t)$ are two sets of generalized coordinates and $Y_i(x)$ is a dimensionless set of eigenfunctions which satisfies the beam end conditions and the following orthogonality conditions:

$$\int_0^\ell Y_i(x) \cdot Y_j(x)\, dx \quad = a_j \text{ for } i=j, \quad = 0 \text{ for } i \neq j \tag{12}$$

$$\int_0^\ell \frac{d^2 Y_i(x)}{dx^2} \cdot Y_j(x)\, dx \quad = b_j \text{ for } i=j, \quad = 0 \text{ for } i \neq j \tag{13}$$

$$\int_0^\ell \frac{d^4 Y_i(x)}{dx^4} \cdot Y_j(x)\, dx \quad = c_j \text{ for } i=j, \quad = 0 \text{ for } i \neq j \tag{14}$$

Substituting Eqs. (10) and (11) in Eq. (9), multiplying the resulting equation throughout by $Y_j(x)$ and integrating with respect to x from 0 to ℓ making use of Eqs. (12)-(14) we get

$$\begin{aligned} &\frac{d^5 T_j}{dt^5} + \alpha\frac{d^4 T_j}{dt^4} + \beta_j\frac{d^3 T_j}{dt^3} + \gamma_j\frac{d^2 T_j}{dt^2} + \lambda_j\frac{dT_j}{dt} + \delta_j T_j \\ &= \theta_j P_j + \xi_j\frac{dP_j}{dt} + \mu\frac{d^2 P_j}{dt^2} + \nu\frac{d^3 P_j}{dt^3} \end{aligned} \tag{15}$$

where

$$\alpha = \frac{1}{c} \tag{16}$$

$$\beta_j = \frac{AkG}{\rho I} - \frac{b_j}{a_j \rho}\left(kG + \frac{E^*}{c}\right) \tag{17}$$

$$\gamma_j = \frac{AkG}{c\rho I} - \frac{b_j}{a_j c\rho}(kG + E) \tag{18}$$

$$\lambda_j = \frac{c_j E^* kG}{a_j c\rho^2} \tag{19}$$

$$\delta_j = \frac{c_j EkG}{a_j c\rho^2} \tag{20}$$

$$\theta_j = \frac{kG}{c\rho^2 I} - \frac{b_j E}{a_j Ac\rho^2} \tag{21}$$

$$\xi_j = \frac{kG}{\rho^2 I} - \frac{b_j E^*}{a_j Ac\rho^2} \tag{22}$$

$$\mu = \frac{1}{cA\rho} \tag{23}$$

$$\upsilon = \frac{1}{A\rho} \tag{24}$$

Now we define the state variables

$$x_{5j-4} = T_j \tag{25a}$$

$$x_{5j-3} = \frac{dT_j}{dt} \tag{25b}$$

$$x_{5j-2} = \frac{d^2 T_j}{dt^2} \tag{25c}$$

$$x_{5j-1} = \frac{d^3 T_j}{dt^3} \tag{25d}$$

$$x_{5j} = \frac{d^4 T_j}{dt^4} \tag{25e}$$

and the input variables

$$u_{4j-3} = P_j \tag{26a}$$

$$u_{4j-2} = \frac{dP_j}{dt} \tag{26b}$$

$$u_{4j-1} = \frac{d^2 P_j}{dt^2} \tag{26c}$$

$$\mu_{4j} = \frac{d^3 P_j}{dt^3} \tag{26d}$$

Using these, (15) may be expressed as the set of first-order ordinary differential equations

$$\dot{x}_{5j-4} = x_{5j-3} \tag{27a}$$

$$\dot{x}_{5j-3} = x_{5j-2} \tag{27b}$$

$$\dot{x}_{5j-2} = x_{5j-1} \tag{27c}$$

$$\dot{x}_{5j-1} = x_{5j} \tag{27d}$$

$$\begin{aligned} \dot{x}_{5j} &= -\delta_j x_{5j-4} - \lambda_j x_{5j-3} - \gamma_j x_{5j-2} - \beta_j x_{5j-1} - \alpha x_{5j} \\ &\quad + \theta_j u_{4j-3} + \xi_j u_{4j-2} + \mu\, u_{4j-1} + \nu u_{4j} \end{aligned} \tag{27e}$$

The beam dynamics can be quite accurately determined by the first r modes where the value of r depends on the particular practical situation. Then we obtain the state model for the beam dynamics as

$$\dot{x} = Ax + Bu \tag{28}$$

where the state vector

$$x = [x_1, x_2, \ldots, x_{5r-4}, x_{5r-3}, x_{5r-2}, x_{5r-1}, x_{5r}]^T \tag{29}$$

the input vector

$$u = [u_1, u_2, \ldots, u_{4r-3}, u_{4r-2}, u_{4r-1}, u_{4r}]^T \tag{30}$$

the open-loop system matrix

$$A = \text{diag}[A_1, A_2, \ldots, A_r]_{5r\times 5r} \tag{31}$$

and the input matrix

$$B = \text{diag}[B_1, B_2, \ldots, B_r]_{5r\times 4r} \tag{32}$$

The diagonal blocks of the matrices A and B are

$$A_j = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\delta_j & -\lambda_j & -\gamma_j & -\beta_j & -\alpha \end{bmatrix}_{5\times 5} \tag{33}$$

$$B_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \theta_j & \xi_j & \mu & \nu \end{bmatrix}_{5\times 4} \tag{34}$$

Point Forces and Point Moments

In our beam model we have allowed for a distributed load only. But it is easy to extend this for point forces and moments which are common in practical applications.

For a point force $F_k(t)$ acting at $\hat{x}_k$ we have

$$f(x,t) = F_k(t) \cdot \delta(\hat{x}_k - x) \tag{35}$$

Now substituting Eq. (11) in Eq. (35), multiplying throughout by $Y_j(x)$ and integrating with respect to x over 0 to ℓ making use of well-known properties of Dirac delta function $\delta(x)$ we get

$$P_j(t) = \frac{Y_j(\hat{x}_k)}{a_j} \cdot F_k(t) \tag{36}$$

Any number of point forces ($k = 1,2,\ldots$) may be incorporated.

For a point moment $M_k(t)$ whose positive sense is taken to be $x \to y$, acting at $\hat{x}_k$, we have

$$f(x,t) = -M_k(t)\frac{d\delta}{dx}(\hat{x}_k - x) \tag{37}$$

Now with a procedure identical to that for point forces we obtain

$$P_j(t) = \frac{Y_j'(\hat{x}_k)}{a_j} \cdot M_k(t) \tag{38}$$

Clearly, any number of point moments may be incorporated.

Linear Dampers and Dynamic Absorbers

One application where the foregoing dynamic analysis becomes useful is in vibration control, using linear dampers and dynamic absorbers, of systems that can be approximated by beams. Such systems include flexible launch vehicles, ships, machine tools, and building structures.

For a linear damper located at $\hat{x}_k$

$$F_k(t) = -C_k\frac{dy}{dt}(\hat{x}_k, t) \tag{39}$$

This, on using Eq. (10), can be expressed as

$$F_k(t) = -C_k[0, Y_1(\hat{x}_k), \ldots, 0, Y_r(\hat{x}_k), 0,0,0]x \tag{40}$$

which, in view of Eqs. (36) and (26), is equivalent to expressing the inputs u_i in terms of state variables x_i.

For a dynamic absorber located at $\hat{x}_k$

$$F_k(t) = -\frac{m_k}{g}\frac{d^2 s_k}{dt^2} \tag{41}$$

Here we define two more state variables

$$\begin{aligned} x_{n+2k-1} &= s_k \\ x_{n+2k} &= \frac{ds_k}{dt} \end{aligned} \tag{42}$$

where $n(=5r)$ is the order of the beam model without dynamic absorbers. Hence we have the additional state equations

$$\dot{x}_{n+2k-1} = x_{n+2k} \tag{43a}$$

$$\dot{x}_{n+2k} = -\frac{g}{m_k}F_k(t) \tag{43b}$$

But

$$F_k(t) = -C_k\frac{d}{dt}[y(\hat{x}_k,t) - s_k(t)] - K_k[y(\hat{x}_k,t) - s_k(t)]$$

This on using Eqs. (10), (25), and (42) becomes

$$F_k(t) = -C_k\Big[\sum_i Y_i(\hat{x}_k)\cdot x_{5i+2} - x_{n+2k}\Big] - K_k\Big[\sum_i Y_i(\hat{x}_k)\cdot x_{5i+1} - x_{n+2k-1}\Big] \tag{44}$$

Here again we have expressed the inputs in terms of the state variables. It is to be noted that each dynamic absorber increases the system order by two.

Viscoelastic and Hysteritic Internal Damping

The constant E^* represents frequency-dependent viscoelastic internal damping which is predominant in many metals. However, it is well known[6] that frequency-independent hysteritic internal damping is also significant in most engineering materials. A method to include the latter ef-

Table 1 Beam natural frequencies

Mode	Natural Frequency (rad/sec)		
	Proposed model		Bernoulli-Euler
	First spectrum	Second spectrum	beam
1	20.6	2.25×10^4	26.3
2	55.9	2.26×10^4	105.2
3	97.3	2.26×10^4	236.6

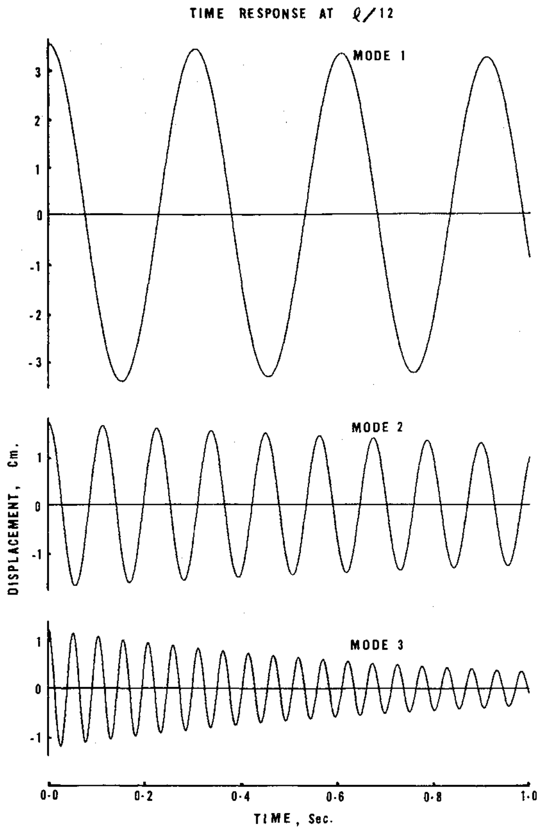

Fig. 1 Modal time responses of the beam.

fects in the proposed beam model is as follows.[5] In each model Eq. (15) for the jth mode of vibration, we replace E^* by

$$E_j^* = g_1 + \frac{g_2}{\omega_j} \qquad (45)$$

Here g_1 represents the viscoelastic damping and g_2 the hysteritic damping. The ω_j is the jth natural frequency of vibration calculated assuming frequency independent E^*.

Numerical Example

We consider a simply supported beam. It is well-known[6] that in this case the eigenfunctions are

$$Y_j(x) = \sin\frac{j\pi x}{\ell}$$

Hence

$$\frac{b_j}{a_j} = -\left(\frac{j\pi}{\ell}\right)^2$$

and

$$\frac{c_j}{a_j} = \left(\frac{j\pi}{\ell}\right)^4$$

An American Standard structural beam with following specifications is chosen.

$E=29\times10^6$	psi	(2.04×10^6	kgf cm $^{-2}$	)
$G=10\times10^6$	psi	($7.03\text{x}10^5$	kgf cm $^{-2}$	)
$E^*=1\times10^4$	psi sec	(703.1	kgf sec cm $^{-2}$	)
$\rho=7.36\text{x}10^{-4}$	lb sec^2 in.$^{-4}$	(802.1	kg sec^{2}m^{-4}	)
$A=9.26$	in.^{2}	(5.97×10^{-3}	m^2	)
$\ell=600.0$	in.	(15.24	m	)
$I=215.8$	in.^{4}	(8.98×10^{-5}	m^4	)
$g=386$	in. sec^{-2}	(9.81	m sec^{-2}	)
$c=0.04$	sec			
$k=0.87$				

The value of E^* is due to Konda and Konno[8] and that of k is due to Cowper.[9] The value of c is selected so that ωc is approximately equal to unity for the first natural frequency.[2]

For a preliminary analysis we consider the first three modes of vibration ($r=3$) and study the free vibration ($\boldsymbol{u}=\boldsymbol{0}$). The negatives of last rows of the diagonal blocks $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ may be computed as,

$$[8.766\times10^{12},\ 3.023\times10^{9},\ 1.272\times10^{10},\ 5.076\times10^{8}, 25.0]$$

$$[1.403\times10^{14},\ 4.836\times10^{10},\ 1.282\times10^{10},\ 5.086\times10^{8}, 25.0]$$

$$[7.100\times10^{14},\ 2.448\times10^{11},\ 1.300\times10^{10},\ 5.102\text{x}10^{8}, 25.0]$$

Natural frequencies of the beam were obtained by computing the eigenvalues of matrix A using an available computer program. These are given in Table 1. A second spectrum of frequencies appears in the present case as for the Timoshenko beam. These frequencies are very high and can be neglected for most practical purposes. Another important observation is that the frequencies of the higher modes of vibration are considerable lower than those predicted by the elementary BE theory. The reason for this becomes evident referring to Eqs. (17) and (18). For large ℓ, the magnitude of (b_j/a_j) is relatively small and the terms independent of j dominate. Hence the contribution from BE theory to the coefficients β_j and γ_j becomes negligible in comparison to that from Timoshenko theory.

Modal time responses are shown in Fig. 1. These were obtained using an available computer routine to solve Eq. (28) by means of a fourth-order Runge-Kutta recursion and an available on-line plotting facility for response plotting. Beam initial conditions used correspond to a load of $3P$ at $\ell/12$ and $-0.9P$ at $\ell/4$ where $P=1000$ lbf (453.6 kgf). The decay of the time responses which increases with mode number, is due to the presence of internal damping.

Conclusions

The proposed model takes into account the important effects due to internal damping of material, shear deformation, and rotatory inertia. In addition the fact that the SLS model is used to represent internal damping makes this beam model valid even for very high frequencies of vibration. The structure of the model is such that it can be handled by available computer packages for eigenvalue and eigenvector computations, solution of the sets of first-order ordinary differential equations with constant coefficients, optimization of inputs such as loads, controls, and their locations with respect to a certain performance criterion etc. A conveninet way to include vibration controls was given. Frequency-independent hysteritic internal damping as well as frequency-dependent viscoelastic damping may be incorporated in the model, by means of the method suggested. Effects of random disturbances on the beam response and the optimal estimation of beam response in the presence of random noise in measurements can be conveniently handled by well-developed stochastic control and estimation theory. Finally, by systematically varying the internal damping and elasticity parameters g_1, g_2, c and E (which reduces to varying elements of the system matrix **A**), and determining the resulting system responses, it is possible to study the associated solid mechanics problem. The simple numerical example given illustrates the use of the proposed beam model in solving an elementary beam vibration problem

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